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MATH 347H: FUNDAMENTAL MATHEMATICS, FALL 2017

PRACTICE PROBLEMS FOR MIDTERM 2

1. **Equivalence relation generated by a collection of sets.** Let X be a set and let $\mathcal{C} \subseteq \mathscr{P}(X)$ be a collection of subsets of X. Define a binary relation $E_{\mathcal{C}}$ on X by
$$xE_{\mathcal{C}}y :\Leftrightarrow \forall S \in \mathcal{C}\,(x \in S \Leftrightarrow y \in S).$$
   (a) Prove that $E_{\mathcal{C}}$ is an equivalence relation.
   (b) Determine $E_{\mathcal{C}}$ explicitly for the trivial cases: $\mathcal{C} = \emptyset$ and $\mathcal{C} = \mathscr{P}(X)$.
   (c) As a concrete example, let $X := \mathbb{R}$ and let $\mathcal{C}$ be the collection of all open intervals with integer endpoints, i.e.
$$\mathcal{C} := \{(n,m) : n,m \in \mathbb{Z},\ n < m\}.$$
   Explicitly describe the equivalence classes of $E_{\mathcal{C}}$.

2. **Equivalence relations induced by functions.**
   (a) For a function $f : X \to Y$, define a binary relation E_f on X by
$$x_0 E_f x_1 :\Leftrightarrow f(x_0) = f(x_1).$$
   Prove that E_f is an equivalence relation. We call it the equivalence relation induced by f.
   (b) Let $E_{\mathbb{Z}}$ be the binary relation on $\mathbb{R}$ defined by $xE_{\mathbb{Z}}y :\Leftrightarrow x - y \in \mathbb{Z}$. Prove that $E_{\mathbb{Z}}$ is an equivalence relation and find a function $f : \mathbb{R} \to [0,1)$ such that $E_{\mathbb{Z}} = E_f$.
   (c) More generally, for any equivalence relation E on a set X, find a set Y and a function $f : X \to Y$ such that $E = E_f$.

   Hint. Quotient by E.

3. Prove that there is no $q \in \mathbb{Q}$ with $q^2 = 3$. Informally speaking, the question asks to prove that $\sqrt{3}$ is not rational.

4. Let x be a symbol for a variable (with no meaning); we call it an *indeterminate variable*. For a commutative ring $(R, +, \cdot, 0_R, 1_R)$, a *polynomial* over R is an expression of the form $a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots\ldots a_1 x + a_0$, where $n \in \mathbb{N}$, for each $i \leq n$, $a_i \in R$, and either $a_n \neq 0_R$ or $n = 0$. If $a_n \neq 0_R$, we say that the *degree* of this polynomial is n; otherwise (i.e. when $n = 0$ and $a_0 = 0_R$), the *degree* is declared $-\infty$.
   (a) Letting $R[x]$ denote the set of all polynomials (of all degrees), define binary operations $+$ and $\cdot$ on $R[X]$ to make it into a ring.
   (b) Prove that $R[x]$ is a domain if and only if R is a domain.

5. Let $F(\mathbb{Q})$ denote the set of all functions $\mathbb{Q} \to \mathbb{R}$, so each element $f \in F(\mathbb{Q})$ is a function from $\mathbb{Q}$ to $\mathbb{R}$. Define a function $\delta_0 : F(\mathbb{Q}) \to \mathbb{R}$ by mapping each $f \in F(\mathbb{Q})$ to its value at 0, i.e. $\delta_0(f) := f(0)$. This function is called the *Dirac distribution* at 0.

(a) Prove that δ_0 is surjective.

(b) Explicitly define two distinct right-inverses for δ_0.

(c) Letting $M_<(\mathbb{Q})$ be the subset of $F(\mathbb{Q})$ of all strictly increasing functions, determine the sets $\delta_0\big(M_<(\mathbb{Q})\big)$ and $\delta_0\big(M_<(\mathbb{Q})^c\big)$.

(d) Determine the set $\delta_0^{-1}(\mathbb{Z})$.

6. Let $F(\mathbb{R})$ denote the set of all functions $\mathbb{R} \to \mathbb{R}$. The composition $f \circ g$ of two functions $f, g \in F(\mathbb{R})$ is a binary operation on $F(\mathbb{R})$. Determine whether

(a) $\circ$ is associative;

(b) $\circ$ is commutative;

(c) there is a $\circ$-identity;

(d) every $f \in F(\mathbb{R})$ has a $\circ$-inverse.

Prove each of your answers. If an answer is negative, provide an explicit counterexample.

7. For a set A, write down all of the equivalent conditions you know for A to be finite (including the definition). Prove that all of them are equivalent to each other.

8. Prove all versions and corollaries of the Pigeonhole Principle on your own.

9. For sets A, B, recall that we write $A \cong B$ to mean that there is a bijection $A \xrightarrow{\sim} B$; in this case, we say that A and B are *equinumerous*. Prove that the following sets are equinumerous with $\mathbb{N}$.

(a) $\mathbb{N}^+$.

(b) The set of all odd numbers natural numbers;

(c) $\mathbb{Z}$;

(d) The set of all integers divisible by 6;

(e) $\mathbb{N}^2$;

(f) $\mathbb{N}^7$.